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Equivalence Theorem for Higher Order Equations

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We show that the theory of an nth-order field equation, minimally coupled to electromagnetism, is completely equivalent to the theory of n independent second-order equations, also minimally coupled to the electromagnetic field. The equivalence is shown to hold as an algebraic identity between the respective matrix elements for a given order of the perturbative solution. A general functional proof is also given. The equivalence shows that the higher order theory is both renormalizable and unitary.

1. INTRODUCTION

We have recently shown that the theory of a field obeying a higher order equation, minimally coupled to the electromagnetic field, is equivalent to that of a set of fields obeying second-order equations of motion (Bollini *et al.*, 1997a, b). The general proof was based on functional methods, and was rather symbolic (or abstract) in character. Here we intend to develop mainly the algebraic aspect of the equivalence.

We take the equation found in Bollini and Giambiagi (1985) when studying supersymmetry in spaces of arbitrary dimensions, namely

$$(\Box^n - m^{2n})\varphi = 0 \tag{1}$$

This equation implies n modes of propagation for the scalar field φ (Schnitzer and Sudarshan, 1961; Barci *et al.*, 1994). Each mode is characterized by a particular mass parameter. In fact, equation (1) can be factorized into n

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Klein–Gordon factors:

$$(\Box^n - m^{2n})\varphi \equiv \prod_{s=1}^{n} (\Box - e_s m^2)\varphi \tag{2}$$

where

$$e_s = e^{(2\pi i/n)(s-1)} \qquad (s = 1, 2, \ldots, n) \tag{3}$$

When we introduce the electromagnetic field by replacing the common derivative with the gauge covariant one, the D'Alembertian $\Box$ is changed into

$$\Box \to \Box' = (\partial_\mu - ieA_\mu)(\partial^\mu - ieA_\mu)$$
$$\Box' = \Box - 2ieA\cdot\partial - e^2A^2 \qquad (\partial_\mu A^\mu = 0) \tag{4}$$

Equation (1) is transformed into

$$(\Box'^n - m^{2n})\varphi = 0 \tag{5}$$

From (4) we get for $\Box'^n$ a polynomial in the charge e:

$$\Box'^n = \Box^n + e\tilde{I}_1^{(n)} + e^2\tilde{I}_1^{(n)} + \cdots + e^{2n}\tilde{I}_{2n}^{(n)}$$
$$= \Box^n + \tilde{I}(eA) \tag{6}$$

With (6), equation (5) takes the form

$$(\Box^n - m^{2n})\varphi = -\tilde{I}(eA)\varphi \tag{7}$$

The field φ obeys an equation of order n in the D'Alembertian operator, with an interaction which is a polynomial of degree $2n$ in the coupling constant.

Our task here will be to examine the consequences of the interaction implied by the right-hand side of equation (7).

Of course, any of the interaction terms $\tilde{I}_l^n$ can be obtained from

$$\Box'^n = (\Box - 2ieA\cdot\partial - e^2A^2)^n \tag{8}$$

The first one comes from a factor $-2ieA\cdot\partial$ and $n-1$ D'Alembertian factors:

$$\tilde{I}_1^{(n)} = -2i(\Box^{n-1}A\cdot\partial + \Box^{n-2}A\cdot\partial\Box + \cdots$$
$$+ \Box A\cdot\partial\Box^{n-2} + A\cdot\partial\Box^{n-1}) \tag{9}$$

When we take the Fourier transform of (9), the derivative operator $-i\partial_\mu$ is transformed into the momentum vector p_μ. The D'Alembertian is transformed into $-p^2$. The vector A_μ leaves its place to the polarization vector ε_μ of the photon. The fact that the momentum p_μ before the emission of the photon is different from the momentum q_μ after the emission is related to the fact that ∂_μ does not commute with A_ν.

Thus the Fourier transform of (9) gives rise to

$$\tilde{I}_1^{(n)} = 2(-1)^{n-1}(p^{2(n-1)}\varepsilon \cdot p + p^{2(n-2)}\varepsilon \cdot pq^2 + \cdots + \varepsilon \cdot pq^{2(n-1)})$$

$$q = p - k, \qquad \varepsilon \cdot p = \varepsilon \cdot q$$

i.e.,

$$\tilde{I}_1^{(n)} = 2(-1)^{n-1}P^{n-1}(p^2, q^2)\varepsilon \cdot p \tag{10}$$

where

$$P^t(x, y) = \sum_{a+b=t} x^a y^b = \sum_{s=0}^{t} x^{(t-s)} y^s \tag{11}$$

The second order in e contains two factors $A \cdot \partial$ and one factor A^2, i.e.,

$$\begin{aligned}\tilde{I}_2^{(n)} = &-(\square^{n-1}A^2 + \square^{n-2}A^2\square + \cdots + A^2\square^{n-1})\\ &-4(\square^{n-2}A \cdot \partial A \cdot \partial + \square^{n-3}A \cdot \partial \square A \cdot \partial\\ &+\square^{n-3}A \cdot \partial A \cdot \partial \square + \cdots + A \cdot \partial A \cdot \partial \square^{n-2})\end{aligned}$$

whose Fourier transform is

$$\begin{aligned}I_2^{(n)} = &-2P^{n-1}(p_1^2, p_2^2)(-1)^{n-1}\varepsilon_1 \cdot \varepsilon_2\\ &+ 4(-1)^{n-2}P^{n-2}(p_1^2, p^2, p_2^2)\varepsilon_1 \cdot p_1\varepsilon_2 \cdot p_2\\ &+4(-1)^{n-2}P^{n-2}(p_1^2, q^2, p_2^2)\varepsilon_1 \cdot p_2\varepsilon_2 \cdot p_1\end{aligned} \tag{12}$$

where

$$p_2 = p_1 + k_1 + k_2; \qquad p = p_1 + k_1; \qquad q = p_1 - k_2$$

$$P^t(x, y, z) = \sum_{a+b+c=t} x^a y^b z^c \tag{13}$$

We can see that the interaction terms give rise to the coefficients $P^t(x_1, x_2, \ldots)$ whose main properties are going to be specified in the next section.

2. VERTEX FACTORS

Each factor $P^t(x_1, \ldots, x_s)$ is a sum over all monomials of degree t, formed with products of powers of its arguments:

$$P^t(x_1, \ldots, x_s) = \sum_{a_1+\cdots+a_s=t} x_1^{a_1} x_2^{a_2} \cdots x_3^{a_s} \tag{14}$$

where all a_l are nonnegative integers.

They have some interesting (and useful) properties. We define

$$P^t = 0 \qquad \text{for} \quad t < 0$$

From (14) we get

$$P^0(x_1, \ldots, x_s) = 1; \qquad P^1(x_1, \ldots, x_s) = \sum_{l=1}^{s} x_l; \qquad P^t(x) = x^t \quad (15)$$

All $P^t(x_1, \ldots, x_s)$ are symmetrical homogeneous functions of their arguments:

$$P^t(x_1, \ldots, x_i, \ldots, x_j, \ldots, x_s) = P^t(x_1, \ldots, x_j, \ldots, x_i, \ldots, x_s) \quad (16)$$

$$P^t(\alpha x_1, \alpha x_2, \ldots, \alpha x_s) = \alpha^t P^t(x_1, x_2, \ldots, x_s) \quad (17)$$

We can also write (14) as

$$P^t(x_1, \ldots, x_s) = \sum_{a_1=0}^{t} x_1^{a_1} \sum_{a_2+\cdots+a_s=t-a_1} x_2^{a_2} \cdots x_s^{a_s}$$

$$P^t(x_1, \ldots, x_s) = \sum_{a=0}^{t} x_1^a P^{t-a}(x_2, \ldots, x_s) \quad (18)$$

Or, more generally,

$$P^t(x_1, \ldots, x_s) = \sum_{a=0}^{t} P^a(x_1, \ldots, x_i) P^{t-a}(x_{i+1}, \ldots, x_s) \quad (19)$$

From (15) we have

$$x_1 P^t(x_1, \ldots, x_s) = \sum_{a=0}^{t} x_1^{a+1} P^a(x_2, \ldots, x_s)$$

$$\sum_{b=1}^{t+1} x_1^b P^{t+1-b}(x_2, \ldots, x_s) = \sum_{b=0}^{t+1} x_1^b P^{t+1-b}(x_2, \ldots, x_s) - P^{t+1}(x_2, \ldots, x_s)$$

$$x_1 P^t(x_1, \ldots, x_s) = P^{t+1}(x_1, x_2, \ldots, x_s) - P^{t+1}(x_2, \ldots, x_s) \quad (20)$$

Also, due to (16),

$$x_2 P^t(x_1, \ldots, x_s) = P^{t+1}(x_1, x_2, \ldots, x_s) - P^{t+1}(x_1, x_3, \ldots, x_s)$$

Then

$$(x_1 - x_2) P^t(x_1, \ldots, x_s) = P^{t+1}(x_1, x_3, \ldots, x_s) - P^{t+1}(x_2, x_3, \ldots, x_s) \quad (21)$$

In particular,

$$(x_1 - x_2)P^t(x_1, x_2) = P^t(x_1) - P^t(x_2) = x_1^t - x_2^t \tag{22}$$

With e_s given by (3), we can write (22) in the form

$$(x_1 - e_s x_2)P^{n-1}(x_1, e_s x_2) = x_1^n - e_s^n x_2^n = x_1^n - x_2^n \qquad (s = 1, \ldots, n) \tag{23}$$

Equation (23) implies that $(x_1 - e_s x_2)$ is a factor of $x_1^n - x_2^n$ for $s = 1, 2, \ldots, n$. Therefore

$$x_1^n - x_2^n = \prod_{s=1}^{n} (x_1 - e_s x_2) \tag{24}$$

According to (11)

$$P^{n-1}(x, e_s y) = \sum_{l=0}^{n-1} x^{n-1-l} e_s^l y^l$$

so that

$$\sum_{s=1}^{n} P^{n-1}(x, e_s y) = \sum_{s=1}^{n} \sum_{l=0}^{n-1} x^{n-1-l} e_s^l y^l$$

But

$$\sum_{s=1}^{n} e_s^l = \sum_{s=1}^{n} e^{(2\pi i l/n)(s-1)} = \frac{1 - e^{2\pi i l}}{1 - e^{2\pi i l/n}} = n\,(\delta_{l,0} + \delta_{l,n} + \delta_{l,2n} + \delta_{l,3n} + \cdots) \tag{25}$$

Then

$$\sum_{s=1}^{n} P^{n-1}(x, e_s y) = nx^{n-1} = P^{n-1}(x, x) \tag{26}$$

We also have

$$\begin{aligned}\sum_{s=1}^{n} P^{n-1}(x, e_s z)P^{n-1}(y, e_s z) &= \sum_{s=1}^{n} \sum_{l=0}^{n-1} x^{n-1-l} e_s^l z^l \sum_{m=0}^{n-1} y^{n-1-m} e_s^m z^m \\ &= \sum_{l=1}^{n} \sum_{m=1}^{n} x^{n-l} y^{n-m} z^{l+m-2} \sum_{s=1}^{n} e_s^{l+m-2}\end{aligned}$$

Taking into account (25), we find

$$\sum_{s=1}^{n} P^{n-1}(x, e_s z)\, P^{n-1}(y, e_s z) = nx^{n-1}y^{n-1} + nz^n P^{n-2}(x, y) \qquad (27)$$

There are similar relations with products of three or more P^{n-1} factors. They will be useful later to prove the equivalence for closed loops (see Section 5). We quote without proof the relation:

$$\sum_{s=1}^{n} P^{n-1}(x, e_s\mu) P^{n-1}(y, e_s\mu) P^{n-1}(z, e_s\mu)$$

$$= \sum_{a,b,c=0}^{n-1} x^{n-1-a}\, y^{n-1-b}\, z^{n-1-c}\, \mu^{a+b+c} n\,(\delta_{a+b+c,0} + \delta_{a+b+c,n}\, \delta_{a+b+c,2n})$$

$$= nx^{n-1}y^{n-1}x^{n-1} + n\mu^n Q(x, y, z) + n\mu^{2n} P^{n-3}(x, y, z) \qquad (28)$$

where

$$Q(x, y, z) = \sum_{a,b,c=0}^{n-1} x^{n-1-a} y^{n-1-b} z^{n-1-c} \delta_{a+b+c,n}$$

Also, it is easy to show that

$$P^t(x, y, z) = \sum_{s=0}^{t} y^{t-s} P^s(x, x) \sum_{s=0}^{t} (s+1) x^s y^{t-s} \qquad (29)$$

and, by repeated use of (20),

$$[P^t(x, y)]^2 = (t+1) P^{2t}(x, y) - \sum_{s=0}^{t-1} (t-s)\,(x^{2t-s} y^s + y^{2t-s} x^s) \qquad (30)$$

3. COMPTON EFFECT

We will now evaluate the matrix element for the Compton effect on a charged particle obeying equation (7). The corresponding propagator obeys

$$(\Box^n - m^{2n})\tilde{G}^{(n)} = i\delta \qquad (31)$$

By taking the Fourier transform of (31) we get

$$G^{(n)} = \frac{(-1)^n i}{p^{2n} - (-m^2)^n} \qquad (32)$$

Of course, for a complete determination of $G^{(n)}$ it is necessary to specify the behavior near the poles. However, we are not going to worry about this point, as (32) is all we need for our purpose (see Sections 6 and 7).

To exhibit explicitly the poles of (32), we use the identity

$$\frac{1}{x^n - a^n} = \frac{1}{na^{n-1}} \sum_{s=1}^{n} \frac{e_s}{x - e_s a}; \qquad e_s = e^{(2\pi i/n)(s-1)} \tag{33}$$

With $x = p^2$ and $a = -m^2$, we get

$$G^{(n)} = \frac{-i}{n - m^{2(n-1)}} \sum_{s=1}^{n} \frac{e_s}{p^2 + e_s m^2} \tag{34}$$

The first term of (34) ($s = 1$) represents the Klein–Gordon propagator. The other terms correspond to the other modes of propagation. The factor $(nm^{2(n-1)})^{-1}$ is the relative normalization of the wave function whose propagator is defined by (31), with respect to that of the usual second-order equation. To obtain an n-independent normalization we have to divide each external line of φ by the factor $(nm^{2(n-1)})^{1/2}$.

We are now ready to evaluate the matrix elements corresponding to any physical process for a higher order equation of the family (5) or (7).

For the Compton effect, the initial and final momenta of the charged bradyon are p_1 and p_2. The incoming (resp. outgoing) photon has polarization ε_1 and momentum k_1 (resp. ε_2 and k_2).

The lowest order Feynman diagrams are shown in Fig. 1.

With the interaction vertices (10) and (12) and the propagator (32), we can write the matrix element:

$$\begin{aligned}
M^{(n)} = {} & [2i(-1)^{n-1}\varepsilon_1 \cdot p_1 P^{n-1}(p_1^2, p^2)] \frac{(-1)^n i}{p^{2n} - (-m^2)^n} \\
& \times [2i(-1)^{n-1}\varepsilon_2 \cdot p_2 P^{n-1}(p^2, p_2^2)] \\
& + [2i(-1)^{n-1}\varepsilon_2 \cdot p_1 P^{n-1}(p_1^2, q^2)] \\
& \times \frac{(-1)^n i}{q^{2n} - (-m^2)^n} [2i(-1)^{n-1}\varepsilon_1 \cdot p_2 P^{n-1}(q^2, p_2^2)] \\
& + i(-1)^n \{4\varepsilon_1 \cdot p_1 \varepsilon_2 \cdot p_2 P^{n-2}(p_1^2, p^2, p_2^2) \\
& + 4\varepsilon_2 \cdot p_1 \varepsilon_1 \cdot p_2 \, P^{n-2}(p_1^2, p^2, p_2^2) \\
& + 2\varepsilon_1 \cdot \varepsilon_2 P^{n-1}\,(p_1^2, p_2^2)\}
\end{aligned}$$

$$\begin{aligned}
M^{(n)} = 4i(-1)^n \Bigg\{ & \varepsilon_1 \cdot p_1 \varepsilon i2 \cdot p_2 \Bigg[\frac{P^{n-1}\,(p_1^2, p^2) P^{n-1}(p^2, p_2^2)}{p^{2n} - (-m^2)^n} \\
& - P^{n-2}\,(p_1^2, p^2, p_2^2) \Bigg]
\end{aligned}$$

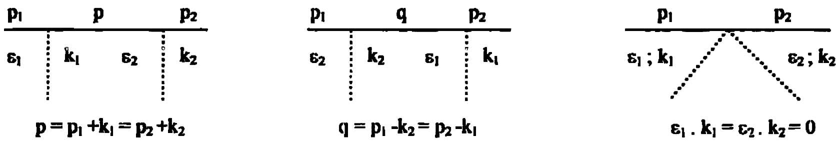

Fig. 1.

$$+ \varepsilon_2 \cdot p_1 \varepsilon_1 \cdot p_2 \left[\frac{P^{n-1}(p_1^2, q^2) P^{n-1}(q^2, p_2^2)}{q^{2n} - (-m^2)^n} - P^{n-2}(p_1^2, q^2, p_2^2) \right] \Bigg\}$$
$$+ 2i(-1)^n \varepsilon_1 \cdot \varepsilon_2 P^{n-1}(p_1^2, p_2^2) \tag{35}$$

Taking $p_1^2 = p_2^2 = -m^2$ in (35), we get, using (22),

$$\begin{aligned} M^{(n)} = 4i(-1)^n \Bigg\{ & \frac{\varepsilon_1 \cdot p_1 \varepsilon_2 \cdot p_2}{p^2 + m^2} [P^{n-1}(-m^2, p^2) \\ & - (p^2 + m^2) P^{n-2}(-m^2, p^2, -m^2)] \\ & + \frac{\varepsilon_2 \cdot p_1 \varepsilon_1 \cdot p_2}{q^2 + m^2} [P^{n-1}(q^2, -m^2) \\ & - (q^2 + m^2) P^{n-2}(-m^2, q^2, -m^2)] \Bigg\} \\ & + 2i(-1)^n \varepsilon_1 \cdot \varepsilon_2 P^{n-1}(-m^2, -m^2) \end{aligned} \tag{36}$$

But, according to (21),

$$\begin{aligned} (x + m^2) P^{n-2}(-m^2, x, -m^2) &= P^{n-1}(x, -m^2) - P^{n-1}(-m^2, -m^2) \\ &= P^{n-1}(x - m^2) + (-1)^n n m^2 \end{aligned}$$

So, we finally obtain for the normalized matrix element $\overline{M}^{(n)} = (nm^{2(n-1)})^{-1} M^{(n)}$

$$\overline{M}^{(n)} = -4i \left(\frac{\varepsilon_1 \cdot p_1 \varepsilon_2 \cdot p_2}{p^2 + m^2} + \frac{\varepsilon_1 \cdot p_2 \varepsilon_2 \cdot p_1}{q^2 + m^2} \right) - 2i \varepsilon_1 \cdot \varepsilon_2 \tag{37}$$

Equation (37) shows the interesting fact that, no matter how high the order of the equation is, we always end up with the matrix element corresponding to the second-order Klein–Gordon equation, coupled to the electromagnetic field (Nishijima, 1969).

It is also possible (and interesting) to answer the following question: What is the amplitude for the Compton effect to produce a transition from the bradyon mode to any other mode of φ?

To answer the question, we take again the matrix element (35). But this time we choose $p_2^2 = -e_s m^2$, where e_s is given by (3).

The new matrix element $M_s^{(n)}$ is proportional to $P^{n-1}(m^2, e_s m^2)$. Using (19), we get

$$P^{n-1}(x, e_s x) = \sum_{l=0}^{n-1} x^{n-1-l} e_s^l x^l = x^{n-1} \sum_{l=0}^{n-1} e_s^l = n x^{n-1} \delta_{s,1} \tag{38}$$

so that

$$M_s^{(n)} = M^{(n)} \delta_{s,1} \tag{39}$$

Equation (34) implies that the probability amplitude for a change from a bradyon mode to any other different mode is exactly zero.

4. DOUBLE PHOTON SCATTERING

For a clear understanding of the algebraic mechanism which shows the equivalence between the higher order theory and the second one, we consider a process involving three external photons. The pertinent Feynman diagrams are shown in Fig. 2.

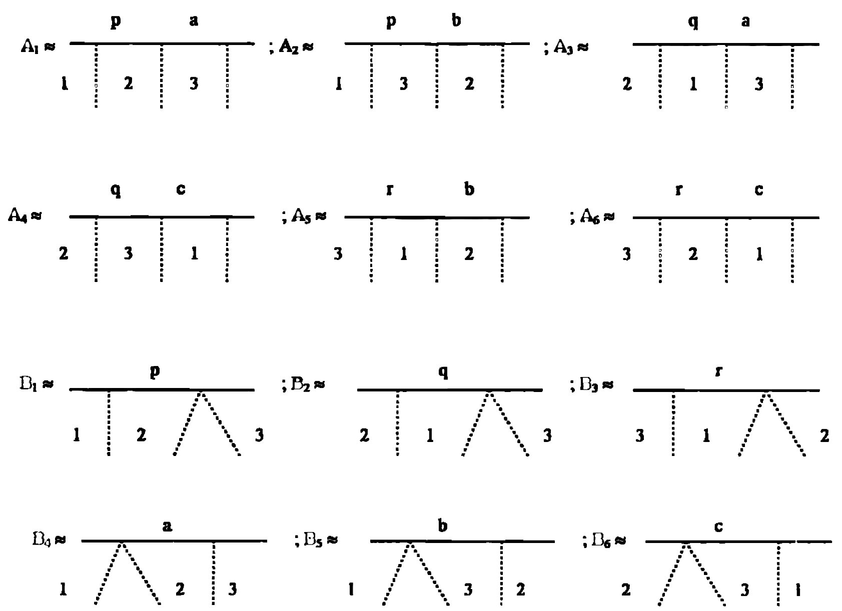

Fig. 2.

We will only write down the matrix elements corresponding to the diagrams A_1, B_1, and C. All other matrix elements follow *mutatis mutandis*. The first- and second-order vertices are explicitly given by (10) and (12). The third-order vertex can be deduced in a similar way:

$$A_1 = 8i(-1)^n \varepsilon_1 \cdot p_1 \varepsilon_2 \cdot p \varepsilon_3 \cdot p_2 \frac{P^{n-1}(p_1^2, p^2) P^{n-1}(p^2, a^2) P^{n-1}(a^2, p_2^2)}{(p^{2n} - (-m^2)^n)(a^{2n} - (-m^2)^n)} \tag{40}$$

$$\begin{aligned} B_1 = {} & 8i(-1)^n \varepsilon_1 \cdot p_1 \frac{P^{n-1}(p_1^2, p^2)}{p^{2n} - (-m^2)^n} \\ & \times [\varepsilon_2 \cdot p \varepsilon_3 \cdot p_2 P^{n-2}(p^2, a^2, p_2^2) + \varepsilon_3 \cdot p \varepsilon_2 \cdot p_2 P^{n-2}(p^2, b^2, p_2^2)] \\ & + 4i(-1)^n \varepsilon_1 \cdot p_1 \varepsilon_2 \cdot \varepsilon_3 \frac{P^{n-1}(p_1^2, p^2) P^{n-1}(p^2, p_2^2)}{p^{2n} - (-m^2)^n} \end{aligned} \tag{41}$$

$$\begin{aligned} C = {} & 8i(-1)^{n-1} \{\varepsilon_1 \cdot p_1 \varepsilon_2 \cdot p \varepsilon_3 \cdot p_2 P^{n-3}(p_1^2, p^2, a^2, p_2^2) \\ & + \varepsilon_1 \cdot p_1 \varepsilon_3 \cdot p \varepsilon_2 \cdot p_2 P^{n-3}(p_1^2, p^2, b^2, p_2^2) \\ & + \varepsilon_2 \cdot p_1 \varepsilon_1 \cdot q \varepsilon_3 \cdot p_2 P^{n-3}(p_1^2, q^2, a^2, p_2^2) \\ & + \varepsilon_2 \cdot p_1 \varepsilon_3 \cdot q \varepsilon_1 \cdot p_2 P^{n-3}(p_1^2, q^2, c^2, p_2^2) \\ & + \varepsilon_3 \cdot p_1 \varepsilon_1 \cdot r \varepsilon_2 \cdot p_2 P^{n-3}(p_1^2, r^2, b^2, p_2^2) \\ & + \varepsilon_3 \cdot p_1 \varepsilon_2 \cdot r \varepsilon_1 \cdot p_2 P^{n-3}(p_1^2, r^2, c^2, p_2^2)\} \\ & + 4i(-1)^{n-1} \{\varepsilon_1 \cdot p_1 \varepsilon_2 \cdot \varepsilon_3 P^{n-2}(p_1^2, p^2, p_2^2) \\ & + \varepsilon_2 \cdot p_1 \varepsilon_1 \cdot \varepsilon_3 P^{n-2}(p_1^2, q^2, p_2^2) + \varepsilon_3 \cdot p_1 \varepsilon_1 \cdot \varepsilon_2 P^{n-2}(p_1^2, r^2, p_2^2) \\ & + \varepsilon_1 \cdot p_2 \varepsilon_2 \cdot \varepsilon_3 P^{n-2}(p_1^2, c^2, p_2^2) + \varepsilon_2 \cdot p_2 \varepsilon_1 \cdot \varepsilon_3 P^{n-2}(p_1^2, b^2, p_2^2) \\ & + \varepsilon_3 \cdot p_2 \varepsilon_1 \cdot \varepsilon_2 P^{n-2}(p_1^2, a^2, p_2^2)\} \end{aligned} \tag{42}$$

For the initial state we will take a normal bradyon ($p_1^2 = -m^2$). The final state may have any of the "masses" of equation (2) ($p_2^2 = -e_s m^2$).

Using (22), we get

$$(x^2 - p_1^2) P^{n-1}(p_1^2, x^2) = x^{2n} - p_1^{2n} = x^{2n} - (-m^2)^n$$

$$(x^2 - p_2^2) P^{n-1}(x^2, p_2^2) = x^{2n} - p_2^{2n} = x^{2n} - (-m^2)^n$$

so that

$$\frac{P^{n-1}(p_1^2, x^2)}{x^{2n} - (-m^2)^n} = \frac{1}{x^2 + m^2}$$

$$\frac{P^{n-1}(x^2, p_2^2)}{x^{2n} - (-m^2)^n} = \frac{1}{x^2 - p_2^2} = \frac{1}{x^2 + e_s m^2} \tag{43}$$

With these simplifications we can write

$$A_1 = 8i(-1)^n \varepsilon_1 \cdot p_1 \varepsilon_2 \cdot p \varepsilon_3 \cdot p_2 \frac{P^{n-1}(p^2, a^2)}{(p^2 + m^2)(a^2 + e_s m^2)} \tag{44}$$

$$\begin{aligned} B_1 = {} & 8i(-1)^n \frac{\varepsilon_1 \cdot p_1}{p^2 + m^2} [\varepsilon_2 \cdot p \varepsilon_3 \cdot p_2 P^{n-2}(p^2, a^2, p_2^2) \\ & + \varepsilon_3 \cdot p \varepsilon_2 \cdot p_2 P^{n-2} (p^2, b^2, p_2^2)] \\ & + 4i(-1)^n \varepsilon_1 \cdot p_1 \varepsilon_2 \cdot \varepsilon_3 \frac{P^{n-1}(p^2, p_2^2)}{p^2 + m^2} \end{aligned} \tag{45}$$

We now gather together the A-matrix elements with similar terms from B and C. For example, A_1 with the first term of B_1, a similar term from B_4, and the first term of C, etc. We have

$$\begin{aligned} A_1' = {} & 8i(-1)^{n-1} \frac{\varepsilon_1 \cdot p_1 \varepsilon_2 \cdot p \varepsilon_3 \cdot p_2}{(p^2 + m^2)(a^2 + c_s m^2)} [P^{n-1} (p^2, a^2) \\ & - (a^2 - p_2^2) P^{n-2}(p^2, a^2, p_2^2) - (p^2 - p_1^2) P^{n-2}(p_1^2, p^2, a^2) \\ & + (a^2 - p_2^2) (p^2 - p_1^2) P^{n-2} (p_1^2, p^2, a^2, p_2^2)] \end{aligned} \tag{46}$$

By using the identity (21), it is not difficult to prove that the bracket in (46) reduces to

$$\begin{aligned} [\cdot] &= P^{n-1}(p_1^2, p_2^2) = P^{n-1}(-m^2, -e_s m^2) = (-m^2)^{n-1} P^{n-1} (1, e_s) \\ &= (-m^2)^{n-1} \sum_{l=0}^{n-1} e_s^l = (-1)^{n-1} n m^{2(n-1)} \delta_{1,s} \end{aligned} \tag{47}$$

[cf. equation (38)].

Finally, when the normalization factor is taken into account, we end up with the n-independent normalized matrix elements

$$\overline{A}_1 = 8i \frac{\varepsilon_1 \cdot p_1 \varepsilon_2 \cdot p \varepsilon_3 \cdot p_2}{(p^2 + m^2)(a^2 + m^2)} \delta_{1,s} \tag{48}$$

$$\overline{B}_1 = -4i \frac{\varepsilon_1 \cdot p_1 \varepsilon_2 \cdot \varepsilon_3}{p^2 + m^2} \delta_{1,s} \tag{49}$$

The presence of the factor $\delta_{1,s}$ in all matrix elements means that the states with masses $e_s m^2$ ($s \neq 1$) cannot be produced by any combination of photons. These "abnormal states" can only exist virtually, associated with internal loops, where they are represented by the respective propagators.

On the other hand, for $s = 1$, equations (48) and (49) represent the matrix elements one would write directly for the double photon scattering on a charged Klein–Gordon particle.

5. VIRTUAL PHOTONS AND CLOSED LOOPS

It is easy to see that virtual photons do not spoil the equivalence we have found in Sections 3 and 4.

Let us take, for example, the production of a photon in the mutual scattering of two charged particles (see Fig. 3).

The first-order vertex to be used at 1, 2, 3, 5, and 6 has the form [compare with (12)]

$$J_1^\mu = (-1)^{n-1} 2ep_a^\mu \, P^{n-1}(p_a^2, p_b^2) \tag{50}$$

where p_a and p_b are (resp.) the momenta of the particle before and after the interaction with the internal photon line. For the vertex 4 we use [compare with (12)]

$$J_2^\mu = (-1)^n 2e^2 \varepsilon^\mu P^{n-1}(p_a^2, p_b^2) + (-1)^n 4e^2 p_a^\mu \varepsilon \cdot p_b P^{n-2}(p_a^2, q^2, p_b^2) + (-1)^n 4e^2 p_b^\mu \varepsilon \cdot p_a P^{n-2}\,(p_a^2, p^2, p_b^2) \tag{51}$$

The line $p_3 \to p_4$ contributes with a factor

$$-2ep_3^\mu P^{n-1}(p_3^2, p_4^2)$$

For $p_3^2 = -m^2$ and $p_4^2 = -e_s m^2$ we have [cf. (47)]

$$P^{n-1}(-m^2, -e_s m^2) = (-1)^{n-1} n m^{2(n-1)} \delta_{s,1}$$

The factor $nm^{2(n-1)}$ disappears after normalization.

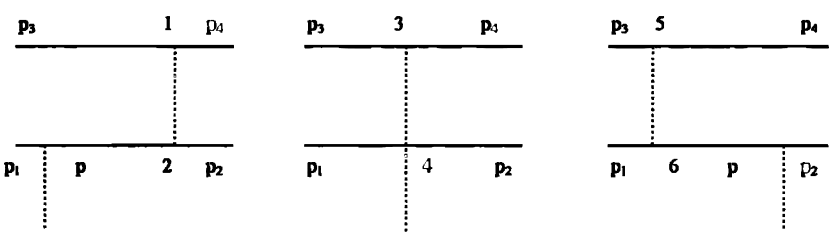

Fig. 3.

The contribution of the line $p_1 \to p_2$ can be treated as a Compton effect in which one of the photons is virtual. The steps shown in Section 3 can then be repeated with only minor changes. The final result is again an n-independent normalized matrix element.

As an example of a closed loop we will first take the second-order photon (polarization) self-energy (Fig. 4).

For diagram A we will use the propagator (32) and the vertex equation (50). (an integration on p_μ is tacit):

$$A = [2i(-1)^{n-1} p_\mu P^{n-1}(p^2, q^2)] \frac{(-1)^n i}{p^{2n} - (-m^2)^n}$$

$$\times [2i(-1)^{n-1}\, p_\nu\, P^{n-1}(p^2, q^2)] \frac{(-1)^n i}{q^{2n} - (-m^2)^n}$$

$$A = 4p_\mu p_\nu \frac{P^{n-1}(p^2, q^2) P^{n-1}(q^2, P^2)}{(p^{2n} - (-m^2)^n)(q^{2n} - (-m^2)^n)} \tag{52}$$

For diagram B we use (32) and (51). We also note that p_μ is an integration variable whose name can be chosen at will. We will take a symmetric expression in p and q:

$$B = (-1)^n 4ip_\mu p_\nu \left[\frac{(-1)^n iP^{n-2}(p^2, q^2, p^2)}{p^{2n} - (-m^2)^n} + \frac{(-1)^n iP^{n-2}(q^2, p^2, q^2)}{q^{2n} - (-m^2)^n}\right]$$

$$+ (-1)^n 2i\eta_{\mu\nu} P^{n-1}(p^2, q^2) \frac{(-1)^n i}{p^{2n} - (-m^2)^n}$$

$$B = -2\eta_{\mu\nu} \frac{P^{n-1}(p^2, p^2)}{p^{2n} - (-m^2)^n}$$

$$- 4p_\mu p_\nu \left[\frac{P^{n-2}(p^2, q^2, p^2)}{p^{2n} - (-m^2)^n} + \frac{P^{n-2}(q^2, p^2, q^2)}{q^{2n} - (-m^2)^n}\right] \tag{53}$$

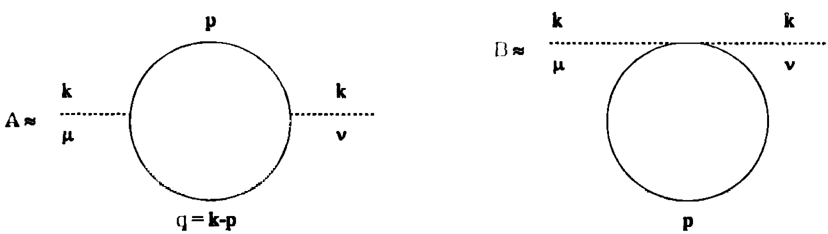

Fig. 4.

Adding together (52) and (53), we obtain

$$A + B = -2\eta_{\mu\nu} \frac{P^{n-1}(p^2, p^2)}{p^{2n} - (-m^2)^n} + \frac{4p_\mu p_\nu}{(p^{2n} - (-m^2)^n)(q^{2n} - (-m^2)^n)} [\cdot] \tag{54}$$

where

$$[\cdot] = (P^{n-1}(p^2, q^2))^2 - (q^{2n} - (-m^2))P^{n-2}(p^2, q^2, p^2) - (p^{2n} - (-m^2))P^{n-2}(q^2, p^2, q^2)$$

But [cf. (29) and (30)]

$$P^{n-2}(p^2, q^2, p^2) + P^{n-2}(q^2, p^2, q^2) = n \sum_{s=0}^{n-2} p^{2s} q^{2(n-2)-2s} = nP^{n-2}(p^2, q^2)$$

$$q^{2n}P^{n-2}(p^2, q^2, p^2) + p^{2n}P^{n-2}(q^2, p^2, q^2) = n \sum_{s=0}^{n-2} (s + 1)(p^{2s}q^{2(n-2)-2s} + q^{2s}p^{2(n-2)-2s})$$

so that

$$[\cdot] = nP^{n-2}(p^2, q^2) - \sum_{s=0}^{n-2} (n - 1 - s)\,(p^{2(2n-2-s)}q^{2s} + q^{2(2n-2-s)}) - \sum_{s=0}^{n-2} (s + 1)(p^{2s}q^{2(2n-2-s)} + q^{2s}p^{2(2n-2-s)}) + n(-m^2)^n P^{n-2}(p^2, q^2)$$

$$= nP^{n-2}\,(p^2, q^2) - n \sum_{s=0}^{n-2} (p^{2s}q^{2(2n-2-s)} + p^{2(2n-2-s)}) + n(-m^2)^n P^{n-2}(p^2, q^2)$$

$$[\cdot] = nP^{2n-2}(p^2, q^2) - nP^{2n-2}(p^2, q^2) + np^{2(n-1)}\, q^{2(n-1)} + n(-m^2)^n P^{n-2}(p^2, q^2)$$

i.e. [cf. equation (27)]

$$[\cdot] = \sum_{s=1}^{n} P^{n-1}(p^2, -e_s m^2)P^{n-1}(q^2, -e_s m^2) \tag{55}$$

Using (55), (43), and (33) (with $a = p^2$ and $x = -m^2$), we can write (54) as

$$A + B = \sum_{s=1}^{n} \left\{ 4p_\mu p_\nu \frac{1}{p^2 + e_s m^2} \frac{1}{q^2 + e_s m^2} - 2\eta_{\mu\nu} \frac{1}{p^2 + e_s m^2} \right\} \tag{56}$$

It can be said that the loops A and B split into n independent loops:

$$A_s + B_s = 4p_\mu p_\nu \frac{1}{p^2 + e_s m^2} \frac{1}{q^2 + e_s m^2} - 2\eta_{\mu\nu} \frac{1}{p^2 + e_s m^2} \qquad (s = 1, 2, \ldots, n) \tag{57}$$

Each of these loops corresponds to a particular second-order mode of propagation and is completely independent of the other modes.

6. FUNCTIONAL PROOF

In the preceding section we have seen an algebraic way to prove the equivalence for a given order of the perturbative expansion of the nth-order equation (5) [or (7)] and a set of n independent Klein–Gordon equations with mass parameters determined by (2). All the scalar fields are minimally coupled to the electromagnetic potential.

However, for the sake of completeness, it is convenient to have a general proof. To this aim we will now introduce functional methods (Faddeev and Slavnov, 1970).

Let us consider a field φ obeying an equation of the form

$$\mathfrak{Q}(\psi)\varphi = 0 \tag{58}$$

where $\mathfrak{Q}(\psi)$ is an operator depending on another field ψ (or set of fields ψ_a).

For example [cf. equation (5)]

$$\mathfrak{Q}(A) = \Box'^n - m^{2n} \tag{59}$$

Let $\mathfrak{Q}$ have an inverse $\mathfrak{Q}^{-1}$, defined by boundary conditions on the solutions of (58).

Let $\mathfrak{X}(\mathfrak{J})$ be the generating functional or partition function related to (58):

$$\mathfrak{X}(\mathfrak{J}) = \int \mathfrak{D}\varphi \exp\left[i \int dx \left(\frac{1}{2} \varphi\mathfrak{Q}\varphi + \mathfrak{J}\varphi \right) \right] \tag{60}$$

$\mathfrak{J}$ is an external source and for the sake of simplicity we are going to ignore the degrees of freedom related to ψ.

The exponent in (60) is a quadratic function of φ. We can then use the functional Gaussian formula (Nishijima, 1969)

$$\int \mathfrak{D}\chi \exp\left[i \int dx \left(\frac{1}{2}\chi\mathcal{P}\chi + \mathcal{J}\chi\right)\right] = \frac{\mathcal{N}'}{|\mathcal{P}|^2} \exp\left[-i \int dx \frac{1}{2} \mathcal{J}\mathcal{P}^{-1}\mathcal{J}\right] \quad (61)$$

where $|\mathcal{P}|$ is the functional determinant of the operator P, and $\mathcal{N}'$ is an irrelevant normalization constant.

Now, let $\mathfrak{Q}$ have the *factorizability* property:

$$\mathfrak{Q} = \mathfrak{Q}_1 \cdot \mathfrak{Q}_2; \qquad [\mathfrak{Q}_1, \mathfrak{Q}_2] = 0 \quad (62)$$

and also the *separability* property:

$$\mathfrak{Q}^{-1} = \alpha\mathfrak{Q}_1^{-1} + \beta\mathfrak{Q}_2^{-1} \quad (63)$$

where α and β are constants.

From (62) it follows that

$$|\mathfrak{Q}| = |\mathfrak{Q}_1| \cdot |\mathfrak{Q}_2| \quad (64)$$

From (60)–(64) we deduce

$$\begin{aligned}\mathfrak{X}(\mathcal{J}) &= \frac{\mathcal{N}'}{|\mathfrak{Q}|^2} \exp\left[-i \int dx \frac{1}{2} \mathcal{J}\mathfrak{Q}^{-1}\mathcal{J}\right] \\ &= \frac{\mathcal{N}'}{|\mathfrak{Q}_1|^2|\mathfrak{Q}_2|^2} \exp\left[-i \int dx \frac{1}{2} \mathcal{J}(\alpha_1\mathfrak{Q}_1^{-1} + \alpha_2\mathfrak{Q}_2^{-1})\mathcal{J}\right] \\ \mathfrak{X}(\mathcal{J}) &= \mathcal{N}' \frac{1}{|\mathfrak{Q}_1|^2} \exp\left[-i \int dx \frac{1}{2} \mathcal{J}\alpha\mathfrak{Q}_1^{-1}\mathcal{J}\right] \\ &\quad \times \frac{1}{|\mathfrak{Q}_2|^2} \exp\left[-i \int dx \frac{1}{2} \mathcal{J}\beta\mathfrak{Q}_2^{-1}\mathcal{J}\right] \end{aligned} \quad (65)$$

We now introduce two independent scalar fields φ_j ($j = 1, 2$) and use (61) in the form

$$\begin{aligned}\int \mathfrak{D}\varphi_j \exp\left[i \int dx \left(\frac{1}{2}\varphi_j\alpha_j^{-1}\mathfrak{Q}_j\varphi_j + \mathcal{J}\varphi_j\right)\right] \\ = \frac{\mathcal{N}_j}{|\mathfrak{Q}_j|^2} \exp\left[-i \int dx \frac{1}{2} \mathcal{J}\alpha_j\mathfrak{Q}_j^{-1}\mathcal{J}\right]\end{aligned} \quad (66)$$

so that from (60) and (65) we get

$$\mathcal{Z}(\mathcal{J}) = \mathcal{N} \int \mathcal{D}\varphi_1 \mathcal{D}\varphi_2$$
$$\times \exp\left[i \int dx \left(\frac{1}{2}\varphi_1\alpha_1^{-1}\mathcal{Q}_1\varphi_1 + \frac{1}{2}\varphi_2\alpha_2^{-1}\mathcal{Q}_2\varphi_2 \right) + \mathcal{J}(\varphi_1 + \varphi_2) \right] \tag{67}$$
$$\mathcal{Z}(\mathcal{J}) = \mathcal{N}\tilde{\mathcal{Z}}(\mathcal{J}) \tag{68}$$

where $\tilde{\mathcal{Z}}$ is the partition functional defined by the right-hand side of (67).

Equation (68) [or (67)] tells us that it is equivalent to say that we have a scalar field obeying (58), or two independent scalar fields φ_j obeying the equations

$$\mathcal{Q}_j\varphi_j = 0; \qquad j = 1, 2 \tag{69}$$

Note that for the equivalence to hold, it is not enough to have the factorizability property (62). It is also necessary that the separability condition (63) be satisfied. As an example we can take (5):

$$\mathcal{Q} = \Box'^n - m^{2n} \tag{70}$$

and the identity (22)

$$(\Box' - m^2)\, P^{n-1}(\Box', m^2) = \Box'^n - m^{2n} \tag{71}$$

Equation (71) is a factorization of (70):

$$\mathcal{Q} = \mathcal{Q}_1 \cdot \mathcal{Q}_2$$
$$\mathcal{Q}_1 = \Box' - m^2; \qquad \mathcal{Q}_2 = P^{n-1}(\Box', m^2)$$

However, it is not possible to write $\mathcal{Q}^{-1}$ as a linear combination of $\mathcal{Q}_1^{-1}$ and $\mathcal{Q}_2^{-1}$ (except for $n = 2$).

Nevertheless, it is easy to see that the equivalence theorem can be extended to the case in which the factorization is multiple, i.e., when we can write

$$\mathcal{Q} = \prod_1^n \mathcal{Q}_j \tag{72}$$

At the same time the separability condition holds in the form

$$\mathcal{Q}^{-1} = \sum_1^n \alpha_j\mathcal{Q}_j^{-1} \tag{73}$$

For (5) we have the factorization (2):

$$\Box'^{n} - m^{2n} = \prod_{s=1}^{n} (\Box' - e_s m^2) \tag{74}$$

and the separability property (33):

$$(\Box'^{n} - m^{2n})^{-1} = \frac{1}{nm^{2(n-1)}} \sum_{s=1}^{n} e_s (\Box' - e_s m^2)^{-1} \tag{75}$$

so that in fact the higher order theory based on (5) is equivalent to n independent second-order theories based on

$$(\Box' - e_s m^2)\varphi_s = 0; \qquad s = 1, 2, \ldots, n \tag{76}$$

7. DISCUSSION

A higher order equation such as (7) implies several modes of propagation for the field φ. The evolution can be treated perturbatively, as a sum of different orders in the coupling constant. There are $2n$ basic couplings with the electromagnetic field [cf. equation (6)], whose form can be determined algebraically. Important parts of these interactions are the vertex coefficients, whose interesting properties are described in Section 2.

The interaction seems to be of the unrenormalizable type, due to the powers of the momenta carried by the P-coefficients. However, the propagator has also a higher power of p^2 in the denominator. By power counting it turns out that the theory is renormalizable. Furthermore, we have shown that the matrix elements corresponding to any process between photons and charged higher order particles can be algebraically reduced to the matrix elements for the interaction between photons and fields obeying second-order equations. This reduction confirms renormalizability and at the same time allows one to established unitarity for the higher order theory.

In fact, the only field to be found asymptotically as a free particle is the bradyon [$s = 1$ in (76)]. All other fields have a half-advanced and half-retarded propagator (Bollini *et al.*, 1994). This propagator was used by Wheeler and Feynman (1945) to describe the electromagnetic interaction in a charged medium which was supposed to be a perfect absorber. Later, the same authors showed that, in spite of the advanced part it contains, this Green function does not contradict causality (Wheeler and Feynman, 1949). Several interesting properties of the Wheeler Green function (including unitarity) are discussed at length in Bollini and Rocca (1997). This function has an on-shell zero. Accordingly, the corresponding fields cannot appear asymptotically in free states. They only appear in closed loops. Furthermore, for each closed loop corresponding to a mass parameter $e_s m^2$ with $s \neq 1$, there is another equal closed loop with the complex conjugate mass parameter $\overline{e_s} m^2$. As a result, the sum of both closed loops contributes with a real function to the amplitude for the process (the Wheeler propagator is a real Green function). Then, all closed loops with $s \neq 1$ give a purely dispersive amplitude. The only absorptive part comes from the bradyon, as it should.

The general proof of Section 6 shows that the equivalence holds for any theory which is both factorizable and separable, i.e., when the equation of motion can be decomposed into two or more factors, and the corresponding propagator can be expressed as a linear combination of the propagators for each of the factors.

It is also clear that practically the same equivalence theorem holds when we consider, instead of a $U(1)$ abelian group, a more general group with the corresponding gauge fields.

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